

KEK-TH-545
KEK Preprint 97-159
September 1997

Comments, Questions and Proposal of a Topological M(atrix) Theory

S.Alam

Theory Group, KEK, Tsukuba, Ibaraki 305, Japan

Abstract

Keeping in mind the several models of M(atrix) theory we attempt to understand the possible structure of the topological M(atrix) theory “underlying” these approaches. In particular we raise the issue about the nature of the structure of the vacuum of the topological M(atrix) theory and how this could be related to the vacuum of the electroweak theory. In doing so we are led to a simple Topological Matrix Model. Moreover it is expected from the current understanding that the noncommutative nature of “spacetime” and background independence should lead to Topological Model. The main purpose of this note is to propose a simple Topological Matrix Model which bears relation to F and M theories. Suggestions on the origin of the chemical potential term appearing in the matrix models are given.

I. INTRODUCTION

By starting with Green-Schwarz action for type IIB superstring and considering its path integral in the Schild gauge Ishibashi et al. [1] proposed a matrix model action,

$$S = -\alpha \left(\frac{1}{4} \text{Tr}([A_\mu, A_\nu]^2) + \frac{1}{2} \text{Tr}(\bar{\psi} \Gamma^\mu [A_\mu, \psi]) \right) + \beta N. \quad (1)$$

ψ is a ten-dimensional Majorana-Weyl spinor field, and A_μ and ψ are $N \times N$ Hermitian matrices. The action in Eq. 1 after dropping the term proportional to N [chemical potential term] constitutes a large- N reduced model of the ten-dimensional super Yang-Mills theory. By noting that N in Eq. 1 is a dynamical variable Fayyazuddin et al. [2] proposed a slightly general form of Eq. 1, viz

$$S = -\alpha \left(\frac{1}{4} \text{Tr}(Y^{-1} [A_\mu, A_\nu]^2) + \frac{1}{2} \text{Tr}(\bar{\psi} \Gamma^\mu [A_\mu, \psi]) \right) + \beta \text{Tr} Y, \quad (2)$$

We note that N in the last term in Eq. 1 is the $N \times N$ identity matrix, i.e. $\text{Tr} I = N$. The positive definite Hermitian matrix Y^{ij} in Eq. 2 is a dynamical variable with its origin in the \sqrt{g} appearing in the Schild action [1]

$$S_{\text{Schild}} = \int d^2\sigma \left(\alpha \sqrt{g} \left(\frac{1}{4} \{X_\mu, X_\nu\}_{PB}^2 - \frac{i}{2} \bar{\psi} \Gamma^\mu \{X_\mu, \psi\}_{PB} \right) + \beta \sqrt{g} \right), \quad (3)$$

The suggested modification of [2] is attractive, for among other things the bosonic part of the classical action Eq. 2 coincides with the Non-Abelian Born-Infeld action after the solution classical equation of motion for the Y -field is substituted back into Eq. 2. We note that the equation of motion for the Y -field is

$$\frac{\alpha}{4} \left(Y^{-1} [A_\mu, A_\nu]^2 Y^{-1} \right)_{ij} + \beta \delta_{ij} = 0, \quad (4)$$

and its solution is

$$Y = \frac{1}{2} \sqrt{\frac{\alpha}{\beta}} \sqrt{-[A_\mu, A_\nu]^2}. \quad (5)$$

The matrix Y^{ab} plays the role of the dynamical variable, the elements of Y can fluctuate while its matrix size is fixed. In contrast the matrix size in the model of [1] is considered as a dynamical variable so that the partition function includes the summation over the matrix size. This summation process is expected to recover the integration over \sqrt{g} [as mentioned earlier] however a proof is not clear.

Earlier Banks et al. [3] proposed/conjectured the matrix model description of M-theory*. Essentially this M(atrrix) theory, as it has been dubbed, is the large N limit of

*The first paper to give the $N=4$ and $N=16$ SUSY gauge quantum mechanics was [5]. The $N=16$ is the precursor to the M(atrrix) theory. I thank M.B.Halpern for pointing reference [5] out to me.

maximally supersymmetric quantum mechanics of $U(N)$ matrices. Some of the standard wisdom about M-theory and related topics is as follows [4]:

- M-theory is the eleven-dimensional theory which after compactification on a circle S^1 can be identified with the ten-dimensional type IIA string theory.
- M-theory is regarded as the strong-coupling limit of type IIA string theory.
- All the known string theories [F-Theory, M-theory, IIA, IIB, the heterotic string based on gauge group $E_8 \times E_8$ and the type I based on the gauge group $SO(32)$] are connected to one another by duality transformations.
- The non-perturbative definition in the Matrix theory of Bank et al., [3] is provided by the D-particle whereas the D-instanton represents the non-perturbative nature of IIB matrix model, [1].
- The conjecture of [3] can be summarized as: M-theory in the Infinite Momentum Frame is a theory with the only dynamical degrees of freedom of D0-branes.

The set-up of this paper is as follows. In the next section we provide motivations via comments and questions for constructing a model based on topological/algebraic arguments that has relation to F-theory and Matrix Model IIB. It also contains conjectures regarding the construction of a model in which the notion of spacetime and noncommutative spacetime arises out of some underlying topological/algebraic structure. Section three contains the actual construction of the desired Topological Matrix Model. In section four we comment on the relation of the Topological Matrix Model to F-theory and Matrix Model IIB. Comments regarding the possible origin of the chemical potential term in Eq. 1 are given in section five. Conclusions and some questions are contained in section six. Finally the appendix contains some known details about the instanton equation in higher dimensions [6].

Recently there has been a lot of interest in Topological Yang-Mills theory in higher dimensions consequently overlap is expected among various works. The work of S. Hirano and M. Kato [7] has overlap with ours[†]. The detailed work of C. Hofman and J-S. Park [8] is also worth citing in this respect. In a forthcoming article we would like to setup an exact comparison between our work and that in [8].

II. EXPECTATIONS FROM THE “UNIQUE” STRING THEORY

We now briefly comment on the expectations from a “unique” string theory:

[†]We thank M. Kato for pointing out their work to us

- Background independence: The appearance of space-time or generation of space-time by the fundamental string theory will be aesthetically and conceptually very pleasing. Why do we want this? Our prejudice [i.e. the reason] for this is what one may call Quantum Mach Principle [QMP]. QMP could simply be stated as: *If there is no “field/matter” there should be no space-time.* As a first step towards realizing the QMP one could consider the noncommutativity of spacetime [9]. In particular it is desirable to obtain a concrete solution to the question as to what structure/principle gives rise to a noncommutative spacetime? At the present there have been several suggestions to represent the noncommutative nature of spacetime by *fuzzy* instantons and D-particles [10,9]. We note that the idea of associating noncommuting matrices with spacetime, in literature, can be traced to recent work of Witten [9]: “The space-time coordinates enter tantalizingly in the formalism as non-commuting matrices.” *Furthermore it is tempting to go even beyond the noncommutative spacetime itself and ask, Can we start from topological space and derive a noncommutative spacetime?* To achieve background independence one needs probably to formulate a topological theory of strings. The matrix model of M-theory and IIB matrix models can be exploited to provide clues to the nature/structure of the Topological Theory of strings.
- Underlying Principle: Just as General Relativity is based on general covariance and the principle of equivalence it is naturally expected that one must decipher or start with a convincing underlying principle for string theory. From the experience with string theory over the last decade or so we expect that the underlying principle should be formulated in the language of topology/algebraic topology constrained by the physical principle of generalized uncertainty principle [i.e. noncommutative spacetime] and spacetime arising out of the existence of “fields” in a *pure vacuum*[‡][i.e. the realization of QMP].
- True Vacuum of String theory: What is the true vacuum of string theory? Moreover how can we select the true vacuum of string nonperturbatively? The electroweak unification had to wait for the discovery of the Higgs mechanism. One may adopt either of the following two strategies to unravel the true vacuum:
 - Try guessing the true vacuum using topological arguments and then construct a string theory around it by taking clues from M and F theories.
 - Construct a truly unified description of string theories and get a strong clue to find the true vacuum.
- Relation between the Standard Model vacuum and the string vacuum: This is important to further our understanding of the electroweak symmetry breaking. We want to

[‡]By pure vacuum we mean a vacuum with no spacetime metric or simply no spacetime structure. The pure vacuum can be conjectured to have properties of a topological space.

know theoretically as to whether the Higgs particle exists or not? If it does exist can we predict its mass from first principles. Moreover we want to know if the Higgs is an elementary or composite particle? If string theory could actually predict the existence [or absence] of Higgs and more importantly come up with the exact prediction for its mass before it is found by the experimentalists, it would certainly be a big boost towards the acceptance of string theory as a description of the real world.

With these motivation in mind we assume for the purposes of this paper that we have a noncommutative spacetime. The coordinates of this noncommutative spacetime are taken to be $N \times N$ matrices X^μ . The index μ takes values from 0 to $D-1$, where D is the dimension of spacetime. The value of D will be fixed in the context of relating the topological matrix model to F-theory and type IIB matrix model, sections three and four. To be particular we take the matrices X^μ to represent “instantons” so that we are naturally led to choose the self-dual equation 13 as our gauge-choice. The spacetime is subject to arbitrary deformations $X^\mu \longrightarrow X^\mu + \Delta X^\mu$. We define the measure

$$|\Delta X|^2 = \text{Tr}[\eta_{\mu\nu} \delta X^\mu \delta X^\nu] \quad (6)$$

$\eta_{\mu\nu}$ is the Minkowski metric with signature $(D-1, 1)$. In keeping with the ADHM description of instantons we take the matrices X^μ to lie in the adjoint representation of the $U(N)$ group.

As pointed out earlier it is tempting to go even beyond the noncommutative spacetime. This would imply defining the “measure” over some topological space and recovering the measure over Minkowski space in Eq. 6 by some suitable reduction procedure. However for the purposes of this paper we adhere to the measure defined in Eq. 6.

III. TOPOLOGICAL MATRIX MODEL

We now turn to give the Topological Matrix Model [TMM]. As is well-known [11,12] we can classify topological field theories into two categories:

- Topological Models with no *explicit* metric dependence. Known examples in this category include three-dimensional Chern-Simons theory and 2+1 gravity.
- Topological models where a metric may be present but varying the background metric does not change the theory i.e. the theory is independent of the metric. This class of theories is called cohomological topological field theories [CTFT]. The metric enters CTFT through BRST gauge fixing and thus the metric is introduced as a gauge artifact. One of the consequences of the metric being a gauge artifact is that the energy momentum tensor in CTFT is BRST trivial. One can see this by noting that the energy momentum tensor [by definition] is given by the variation of the Lagrangian with respect to the metric

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta L}{\delta g^{\mu\nu}}$$

and since the gauged-fixed action with its Faddeev-Popov term can be written as $\{Q, F\}$ for some field F , the energy momentum tensor can be written as

$$T_{\mu\nu} = \{Q, F_{\mu\nu}\}.$$

One procedure of constructing cohomological theories is to postulate a gauge transformation under which the original action is invariant. The original action is taken to be zero or a pure topological quantity. One then gets a Gauge-Fixed [GF] action written as a BRST variation.

We now start with zero action in the usual manner [11]

$$L = 0 \tag{7}$$

and construct cohomological model. We recall that when considering Topological Yang-Mills symmetry one considers the infinitesimal transformations [13]:

$$\delta A_\mu = D_\mu \varepsilon + \varepsilon_\mu, \tag{8}$$

where $A = A_\mu dx^\mu$ is the Yang-Mills field and D_μ is the covariant derivative

$$D_\mu \equiv \partial_\mu + [A_\mu, \quad], \tag{9}$$

ε is the usual Yang Mills local parameter and ε_μ is a *new local 1-form infinitesimal parameter*. The action of δ on the field-strength i.e. the two-form $F = dA + AA$ is

$$\delta F_{\mu\nu} = D_{[\mu} \varepsilon_{\nu]} - [\varepsilon, F_{\mu\nu}]. \tag{10}$$

In lieu of our discussion in the section two [in particular the last part of this section] and Eq. 8 we subject the non-commutative coordinates to arbitrary deformations and assume that

$$\delta X^\mu = \varepsilon^\mu, \tag{11}$$

where ε^μ are $N \times N$ matrices.

The zero action 7 is assumed to be invariant under the gauge transformation

$$\delta_1 X_\mu = \psi_\mu. \tag{12}$$

Next we choose a gauge so that $[X_\mu, X_\nu]$ is self-dual [Appendix],

$$\begin{aligned} \lambda[X_\mu, X_\nu] &= \frac{1}{2} S_{\mu\nu\alpha\beta} [X^\alpha, X^\beta], \\ F_{\mu\nu} &\equiv i[X_\mu, X_\nu], \\ \lambda F_{\mu\nu} &\equiv \frac{1}{2} S_{\mu\nu\alpha\beta} F^{\alpha\beta}. \end{aligned} \tag{13}$$

in view of the motivations explained earlier. λ is the “eigenvalue” in the self-dual equation [Appendix]. There has been a lot of interest in “instanton” equation, especially recently, [14,15]. To this end we have included in the Appendix some relevant details/formulae about the higher dimensional “instanton” equation [6].

Applying the quantization procedure to the zero action subject to Eqs. 7 and keeping in mind the full BRST transformation laws including Eq. 12, viz,

$$\begin{aligned}\delta_1 X_\mu &= \psi_\mu, \\ \delta_1 \chi_{\mu\nu} &= iB_{\mu\nu}, \\ \delta_1 B_{\mu\nu} &= 0, \\ \delta_1 \psi_\mu &= 0,\end{aligned}\tag{14}$$

we may write the gauge fixed action with Faddeev-Popov [FP]

$$L_{\text{GF+FP}}^1 = \text{Tr} \left(\frac{1}{4} B^{\mu\nu} [\lambda F_{\mu\nu} - \frac{1}{2} S_{\mu\nu\alpha\beta} F^{\alpha\beta}] - \chi_{\mu\nu} [X^{[\mu}, \psi^{\nu]}] + \frac{1}{8} a B^{\mu\nu} B_{\mu\nu} \right) \tag{15}$$

where $\chi_{\mu\nu}$ and ψ_μ are the FP ghostfields, $B_{\mu\nu}$ is a self-dual auxiliary field, and a is a parameter which takes on in general a different value for each component. For example $a = a_{09}$ when $\mu = 0$ and $\nu = 9$ in Eq. 15.

We have used the subscript 1 for the BRST variation δ to emphasize that we are carrying the quantization [i.e. gauge-fixing procedure] at the first stage, in anticipation that due to hidden symmetry of the gauge-fixed action Eq. 15 we need to repeat the gauge-fixing procedure.

We can write the action in Eq. 15 as a BRST variation using the BRST transformation laws given in Eq. 14,

$$L_{\text{GF+FP}}^1 = -\frac{i}{4} \text{Tr} \left(\delta_1 (\chi^{\mu\nu} [\lambda F_{\mu\nu} - \frac{1}{2} S_{\mu\nu\alpha\beta} F^{\alpha\beta} + \frac{1}{2} a B_{\mu\nu}]) \right). \tag{16}$$

As a check we explicitly act with δ_1 in Eq. 16 and using Eq. 14 we arrive at

$$L_{\text{GF+FP}}^1 = \text{Tr} \left(\frac{1}{4} B^{\mu\nu} [\lambda F_{\mu\nu} - \frac{1}{2} S_{\mu\nu\alpha\beta} F^{\alpha\beta}] - \chi_{\mu\nu} [X^{[\mu}, \psi^{\nu]}] + \frac{1}{8} a B^{\mu\nu} B_{\mu\nu} \right) \tag{17}$$

which is nothing but Eq. 15.

Next we move to the second stage of gauge-fixing. Since ψ^μ has a ghost-symmetry this can be parametrized by the ghost field Φ , namely $\delta_2 \psi_\mu = [X_\mu, \Phi]$. Moreover the action above Eqs. 16, 17 possesses a hidden symmetry, $\delta_2 \psi_\mu = [X_\mu, \Phi]$, and $\delta_2 B_{\mu\nu} = ie[\Phi, \chi_{\mu\nu}]$ where e is a constant. We must thus continue the quantization procedure by fixing this symmetry. To this end introduce a set of fields Φ , $\bar{\Phi}$ and η . Keeping these points in mind the set of BRST transformations reads

$$\begin{aligned}\delta_2 X_\mu &= 0, \\ \delta_2 \chi_{\mu\nu} &= 0,\end{aligned}$$

$$\begin{aligned}
\delta_2 B_{\mu\nu} &= ie[\Phi, \chi_{\mu\nu}], \\
\delta_2 \psi_\mu &= [X_\mu, \Phi], \\
\delta_1 \Phi &= 0, \\
\delta_2 \Phi &= 0, \\
\delta_1 \bar{\Phi} &= 0, \\
\delta_2 \bar{\Phi} &= 2\eta, \\
\delta_1 \eta &= 0, \\
\delta_2 \eta &= -\frac{1}{2}e[\Phi, \bar{\Phi}].
\end{aligned} \tag{18}$$

We have used the subscript 2 for the BRST variation δ to emphasize that we are carrying the quantization [i.e. gauge-fixing procedure] at the second stage.

The gauge-fixed action subject to the BRST rules in Eq. 18 can be written as,

$$L_{\text{GF+FP}}^2 = \text{Tr} \left([\delta_1 + \delta_2] \left(-\frac{1}{2} \bar{\Phi} [X_\mu, \psi^\mu] + \frac{1}{2} s e \bar{\Phi} [\Phi, \eta] + \frac{i}{4} \chi^{\mu\nu} B_{\mu\nu} \right) \right). \tag{19}$$

where s is some parameter.

Carrying out explicitly the action of the BRST variation $\delta_1 + \delta_2$ in Eq. 19 we have,

$$\begin{aligned}
L_{\text{GF+FP}}^2 &= \text{Tr} \left(-\eta [X_\mu, \psi^\mu] - \frac{1}{2} \bar{\Phi} [\psi_\mu, \psi^\mu] \right. \\
&\quad + \frac{1}{2} [X_\mu, \Phi] [X^\mu, \bar{\Phi}] + s e \Phi [\eta, \eta] \\
&\quad + \frac{1}{4} s e^2 [\Phi, \bar{\Phi}]^2 \\
&\quad \left. - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} e \Phi [\chi^{\mu\nu}, \chi_{\mu\nu}] \right).
\end{aligned} \tag{20}$$

We note that the field Φ is unaffected by the BRST variation $\delta_1 + \delta_2$. This implies that the action Eq. 20 is not unique for we can add to it a BRST variation of some fields that give a total contribution of zero, for example we can add to the action some arbitrary collection of the ϕ field which is unaffected by the BRST variation.

The full action is the sum of the two actions Eqs. 15, 20, viz,

$$L_{\text{GF+FP}} = L_{\text{GF+FP}}^1 + L_{\text{GF+FP}}^2. \tag{21}$$

In anticipation of comparison of TMM to other models, we now choose the value of D to be 10. This choice is also guided by the observation that the special properties of γ matrices in eight-dimensions don't recur in higher dimensions [6]. We thus choose the self-dual equation of Eq. 13

$$\lambda[X_\mu, X_\nu] = \frac{1}{2} S_{\mu\nu\alpha\beta} [X^\alpha, X^\beta] \tag{22}$$

to be valid in $D = 10$ and define the totally antisymmetric tensor $S_{\mu\nu\alpha\beta}$ in analogy with Eq. A5 in the Appendix, namely,

$$S^{\mu\nu\alpha\beta} = \xi^T \Gamma^{\mu\nu\alpha\beta} \xi. \quad (23)$$

We demand $\xi^T \xi = 1$ [Appendix]. ξ is a constant spinor, and $\Gamma^{\mu\nu\alpha\beta}$ is the totally antisymmetric product of Γ matrices for $SO(9, 1)$ spinor representation. Since we want to impose the “unique” [6] conditions Eq. A4 arising from the octonionic structure we decompose Γ^μ of $D=10$ in terms of γ^i of eight-dimensional $SO(8)$ such that

$$\begin{aligned} F_{0i} &= 0, \\ F_{9i} &= 0, \\ F_{09} &= 0, i = 1, 2, \dots, 8, \end{aligned} \quad (24)$$

and Eq. A4 holds. Under these conditions $\lambda = 1$ in Eq. 22. The breakdown of Γ^μ in terms of γ^i and the values of antisymmetric tensor which ensures Eq. 24 are as follows:

$$\begin{aligned} \Gamma^0 &= i\sigma_2 \otimes 1, \\ \Gamma^9 &= \sigma_1 \otimes \gamma^9, \\ \Gamma^i &= \sigma_1 \otimes \gamma^i, \\ S^{0ijk} &= 0, \\ S^{9ijk} &= 0, \\ S^{09ij} &= 0, \\ S^{ijkl} &= \xi^T \gamma^{ijkl} \xi, \quad i = 1, 2, \dots, 8. \end{aligned} \quad (25)$$

Keeping in mind the information outlined the total gauge-fixed action Eq. 21 can be written after integrating over the auxiliary field B_{ij} as

$$\begin{aligned} L_{\text{GF+FP}} &= \text{Tr} \left(\frac{1}{4} F_{ij} F^{ij} + a \frac{1}{4} F_{ij} S^{ijkl} F_{kl} \right. \\ &\quad \left. - \frac{1}{8} F_{ij} S^{ijkl} F_{kl} - \chi_{ij} [X^i, \psi^j] \right. \\ &\quad \left. - \eta [X_i, \psi^i] - \frac{1}{2} \overline{\Phi} [\psi_i, \psi^i] \right. \\ &\quad \left. + \frac{1}{2} [X_i, \Phi] [X^i, \overline{\Phi}] + s e \Phi [\eta, \eta] \right. \\ &\quad \left. + \frac{1}{4} s e^2 [\Phi, \overline{\Phi}]^2 - \frac{1}{4} e \Phi [\chi^{ij}, \chi_{ij}] \right). \end{aligned} \quad (26)$$

The set of bosonic fields in Eq. 26 is $(X^i, \Phi, \overline{\Phi})$ where the fermionic set is $(\psi^i, \chi^{ij}, \eta)$. The action in Eq. 26 is now in the form to be compared to the supersymmetric reduced model.

We next choose the value of D to be 9, so that we are starting with $SO(8, 1)$. In order to exploit the “instanton” equation in 7 Euclidean dimension [Appendix] we consider

$SO(8, 1)$ broken into $SO(1, 1) \otimes SO(7)$. Further the subgroup of $SO(7)$ which respects the octonion structure [6] is G_2 . The gauge conditions in this case are obtained by replacing 9 by 8 in 24, letting i run from 1 to 7,

$$\begin{aligned} F_{0i} &= 0, \\ F_{8i} &= 0, \\ F_{08} &= 0, i = 1, 2, \dots, 7, \end{aligned} \tag{27}$$

and deleting the terms with subscript 8 in Eq. A4, obtaining the set of seven equations given in Eq. A15. We note that under these conditions $\lambda = 1$ in Eq. 22.

When the value of D is set to 8, in a like manner we start with $SO(7, 1)$ and consider $SO(7, 1)$ broken into $SO(1, 1) \otimes SO(6)$. The relevant subgroup in this case is of $SO(6)$ is $SU(3) \otimes U(1)/Z_3$ [6] [Appendix]. The gauge conditions in this case are obtained by replacing 8 by 7 in 27 and letting i run from 1 to 6, namely

$$\begin{aligned} F_{0i} &= 0, \\ F_{7i} &= 0, \\ F_{07} &= 0, i = 1, 2, \dots, 6, \end{aligned} \tag{28}$$

and deleting the terms with subscript 7 in Eq. A15, obtaining the set of seven equations given in Eq. A17. Of course under these conditions $\lambda = 1$ in Eq. 22. We note that there are two subgroups of $SO(6)$ which allow an invariant construction of the fourth rank tensor $S_{\mu\nu\alpha\beta}$ namely $SO(4) \otimes SO(2)$ and $SU(3) \otimes U(1)/Z_3$ [6]. The choice $SO(4) \otimes SO(2)$ corresponds to the case where the six dimensional manifold is a direct product of a four dimensional and a two dimensional manifold. The second subgroup $SU(3) \otimes U(1)/Z_3$ is the holonomy group of six dimensional Kahler manifolds.

IV. RELATIONSHIP BETWEEN TMM & OTHER STRING MODELS

In this section we look at the relationship between TMM and other string theories. In particular we want to see the relationship between TMM and F-Theory [16], TMM and matrix model of M-theory and TMM and the matrix model of type IIB string theory [1].

If one were to ask for a model, based on D-dimensional Yang-Mills type, to be written on purely intuitive ground, the action

$$S = -\alpha \left(\frac{1}{4} Tr([A_\mu, A_\nu]^2) \right), \tag{29}$$

would come to mind, where A_μ is the Yang-Mills field. Further insistence on incorporating supersymmetry would lead us to the modified action

$$S_{SRM} = -\alpha \left(\frac{1}{4} Tr([A_\mu, A_\nu]^2) + \frac{1}{2} Tr(\bar{\psi} \Gamma^\mu [A_\mu, \psi]) \right), \tag{30}$$

where ψ is a ten-dimensional Majorana-Weyl spinor field. Eq. 30 is nothing but the action for supersymmetric reduced model [1] and is the same as the action in Eq. 1 without the βN term. The action in Eq. 30 is called supersymmetric reduced model [SRM]. The ten dimensional Super Yang-Mills action i.e. SRM of Eq. 30 can be rewritten in terms of octonions of eight-dimensional space [Appendix] as

$$\begin{aligned}
S_{SRM} = & \text{Tr} \left(-\frac{1}{4}([A_i, A_j]^2) + \frac{1}{2}([A_i, A_0 + A_9][A^i, A^0 - A^9]) \right. \\
& + \frac{1}{8}([A_0 + A_9, A_0 - A_9]^2) \\
& - \frac{1}{2}\lambda_a^L(2[A^{[8}, \lambda_R^{a]}] + c_{abc}[A^{[b}, \lambda_R^{c]}) - \lambda_L^8[A_i, \lambda_R^i] \\
& \left. - \frac{1}{2}(A_0 - A_9)[\lambda_i^R, \lambda_R^i] - \frac{1}{2}(A_0 + A_9)[\lambda_L^8, \lambda_L^8] - \frac{1}{2}(A_0 + A_9)[\lambda_a^L, \lambda_a^L] \right) \quad (31)
\end{aligned}$$

where the indices $i, j = 1, \dots, 8$ and $a, b, c = 1, \dots, 7$ as in the Appendix. We note that the L, R appearing as subscript or superscript denote the chirality of the $SO(8)$ chiral spinors λ . We have written the action of SRM in the form displayed in Eq. 31 to facilitate comparison with the TMM of the previous section. To the same end we have dropped the coupling α . It is straightforward to see that the action of Eq. 31 is the same as the action of TMM in $D = 10$ viz Eq. 26, if one lets

$$\begin{aligned}
X^i & \Longleftrightarrow A^i, \\
\Phi & \Longleftrightarrow A_0 + A_9, \\
\bar{\Phi} & \Longleftrightarrow A_0 - A_9, \\
\psi^i & \Longleftrightarrow \lambda_R^i, \\
2\chi^{8a} & \Longleftrightarrow \lambda_L^a, \\
\eta & \Longleftrightarrow \lambda_L^8,
\end{aligned} \quad (32)$$

except for the term $\text{Tr}(a\frac{1}{4}F_{ij}S^{ijkl}F_{kl} - \frac{1}{8}F_{ij}S^{ijkl}F_{kl})^\S$.

In order to make connection of the TMM with F-Theory we note that F-Theory is formulated in 12 dimensions [16] and is supposed to be the underlying theory of type IIB strings. More precisely F-Theory is defined only through the compactifications on elliptically fibered complex manifolds. For the purposes of this paper we compare the TMM with F-theory in a naive manner ignoring for the moment the task of compactifying TMM according to complicated compactification schemes, such as, for example compactification on K_3 orbifold and T^4/Z_2 .

[§]The simplest gauge choice is to take $a = 0$. The trace of $F_{ij}S^{ijkl}F_{kl}$ for finite N vanishes due to Jacobi-identity and the cyclic property of trace. In large N limit this term may survive and could play a role in the dynamics of matrix model. We do not understand the implications of this term on the dynamics of the matrix model.

If we look at the bosonic content of TMM we can interpret the emergence of Φ and $\overline{\Phi}$ as two “extra coordinates”. We see that besides the 10 dimensional spacetime we have to content with two “extra dimensions” $\Phi + \overline{\Phi}$ and $\Phi - \overline{\Phi}$. More precisely in addition to the eight transverse coordinates X^i [$i = 1, 2, \dots, 8$.] we have the light-cone coordinates X^0 and X^9 and two extra transverse coordinates $\Phi + \overline{\Phi}$ and $\Phi - \overline{\Phi}$. If we write the contribution of these 12 coordinates to explicitly show the signature we have (1, 1), (8, 0) and (1, 1) coming from X^0 and X^9 , X^i and $\Phi + \overline{\Phi}$ and $\Phi - \overline{\Phi}$ respectively. Thus the TMM has 10+2 spacetime dimensions. The lightcone TMM seems to correspond to lightcone F-theory with 9+1=10 transverse coordinates.

It is known [16] that if we compactify F-theory on (1,1) space we will obtain type IIB string theory. Thus if we compactify the (1,1) space i.e. $(\Phi + \overline{\Phi}, \Phi - \overline{\Phi})$ we will obtain a matrix description of the light-cone type IIB theory. Thus a matrix description of the light-cone type IIB theory can be taken to be the SRM on $T^{1,1}$ torus with Φ and $\overline{\Phi}$ directions compactified.

We now turn to the cases of $D=9$ and $D=8$. Let us begin with the $D = 9$ case, where we start with the 9 dimensional spacetime. Taking into account the two extra dimensions [see discussion above], the TMM has 9+2 spacetime dimensions. If we compactify Φ and $\overline{\Phi}$ directions to obtain the $T^{1,1}$ torus, we may write $R^{9,2} \rightarrow R^{8,1} \times T^{1,1}$. Now in M-theory $R^{10,1} \rightarrow R^{9,1} \times S^1$ [3]. If we compactify $R^{9,1}$ on two torus T^2 , we obtain $R^{10,1} \rightarrow (R^{7,1} \times S^1) \times T^2$. At this point we recall that the ‘fundamental’ excitations of M-theory *ala* Banks et al. [3] are 0-branes. In the present model the basic objects are ‘instantons’ [-1-branes]. We conjecture/expect that the TMM model compactified on two-torus, namely, $R^{9,2} \rightarrow R^{8,1} \times T^{1,1}$ is equivalent to M-theory $R^{10,1} \rightarrow (R^{7,1} \times S^1) \times T^2$. For the case of the 8 dimensional spacetime, when we compactify our theory on two-torus we can write $R^{8,2} \rightarrow R^{7,1} \times T^{1,1}$. We may obtain this case by compactifying its higher dimensional counterparts. In the above compactification schemes we have used the simple conventional logic. However as pointed out in [21] the conventional logic leads one to expect that if M-theory is compactified on a two-torus than since 11-2=9 one should obtain a 9 dimensional theory but one finds that if the area of the torus is shrunk to zero the result is 10 dimensional IIB string theory. It would be interesting to examine if we can manufacture an extra dimension for the above case of $R^{8,2} \rightarrow R^{7,1} \times T^{1,1}$.

Finally we comment on the question: What can one say in the context of TMM about the emergence of commutative spacetime and general coordinate transformations? We can write

$$\begin{aligned} X^i &\longrightarrow X^i + \delta X^i, \\ \Phi &\longrightarrow \Phi + \delta \Phi, \\ \overline{\Phi} &\longrightarrow \overline{\Phi} + \delta \overline{\Phi} \end{aligned} \tag{33}$$

for the transformations of the bosonic fields. Since Φ is unchanged under the transformations, as mentioned earlier, see Eq. 18, we can ignore the transformation relation of Φ in Eq. 33. The machinery of recovering the commutative spacetime of the observed world from the noncommutative one is not built in the present topological model. Thus for the present

we assume a background in which the matrices are commuting and replace quantities in Eq. 33 by their commuting counterparts [i.e $X^i \rightarrow x^i$, $\delta X^i \rightarrow g^i(x^i, \phi, \bar{\phi})$ $\Phi \rightarrow \phi$, $\bar{\Phi} \rightarrow \bar{\phi}$, $\delta \bar{\Phi} \rightarrow g^{\bar{\phi}}(x^i, \phi, \bar{\phi})$ where the background fields x^i , ϕ and $\bar{\phi}$ are all mutually commuting] then Eq. 33 takes the form of the general coordinate transformations.

V. THE CHEMICAL POTENTIAL TERM

We note that the action given in Eq. 30 does not contain the chemical potential term. In this section we want to address the question: How does one account for the the chemical potential term, βN term in Eq. 1? We now give some brief comments/conjectures regarding the origin of the chemical potential term in Eq. 1. To this end we recall that this term may be traced back to the \sqrt{g} appearing in the Schild action.

- The term $\sqrt{g}d^2\sigma$ appearing in Schild action is nothing but the area term. The question thus arises if we can *consider this term as arising from the area preserving diffeomorphisms of F and M theories*. Indeed it has been recently claimed by Sugawara [22] that his F theory and M theory can be formulated as gauge theories of area preserving diffeomorphisms algebra. We note that the M-theory of Sugawara [22] is 1-brane formulation rather than the 0-brane formulation of Banks et al. [3] and the F-theory of Sugawara [22] is 1-brane formulation rather than the -1-brane formulation of Ishibashi et al. [1]. Assuming that the reverse of Sugawara's suggestion is true, we can regard the area term as arising from those diffeomorphisms of F and M theories which preserve the area.
- In view of formulating a generalized uncertainty principle for string theories and keeping in mind the work of Yoneya [18] we may regard the area term to be connected with the generalized uncertainty principle.
- It is well-known from the context of cohomological topological field theories [11] that the action is not unique in the sense that we can add to it a BRST variation of some arbitrary collection of fields. We may regard the βN term [Eq. 1] as representing that set of $[X^\mu, X^\nu]$ which are proportional to the identity.
- The chemical potential term could arise from a term which comes from the BRST breaking.

We expect the above approaches to the determination of the origin of the chemical potential term to be interlated or equivalent.

VI. CONCLUSIONS

We have obtained a simple Topological Matrix Model. This model may be considered as a first step in an attempt to:

- Formulate a theory underlying the several known string theories.
- Construct a theory which has background independence built into it.
- Understand the true vacuum of string theory.
- Formulate a theory in which spacetime is a derived concept. It is conjectured that the primordial vacuum has a topological structure without a metric. The metric is expected to arise by quantum topological fluctuation.
- Attempt to understand: The details of how the electroweak vacuum can be accounted for in terms of the true string vacuum.

In conclusion, by construction TMM has a strong semblance to the SRM. It can be related to the F-theory and type IIB matrix model. This is not surprising since it is known or expected that the some topological model is most likely to provide an underlying theory of strings. By construction and by direct comparison [Eqs. 31 & 26] TMM can be regarded to be quite similar to the topologically twisted form of the supersymmetric reduced model.

It would be useful to examine the following questions in an attempt to formulate a fundamental unified theory of strings:

- Can we understand Matrix Models in terms of Knots? [17]
- Is there a Generalized Uncertainty Principle in context in context of strings? [18]
- What formalism of strings is best suited to identify and describe the true vacuum of string theory?

ACKNOWLEDGEMENTS

The author would like to thank H. Kawai, and N. Ishibashi for several discussions and M. Kato for explaining his work. The work of the authors is supported by JSPS fellowship of the Japanese Ministry of Education, Science and Culture [MONBUSHO].

Note Added:

We note that topological theories underlying quantum mechanics on instanton moduli spaces in matrix context are first discussed in a recent [interesting] paper by S. Gukov [23]. This was pointed out to me by S.Gukov after the submission of this work and I thank him for this.

APPENDIX:

As is well-known "instantons" are self-dual solutions of Yang-Mills equation in the compactified Euclidean 4-space [19,20].

The instanton equation can formally be written in D^{**} dimensions as

$$\lambda F^{\mu\nu} = \frac{1}{2} S^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (A1)$$

where λ is a non-zero constant and $T^{\mu\nu\alpha\beta}$ is a totally antisymmetric tensor.

For $D=4$ $S^{\mu\nu\alpha\beta}$ is unique and as can be readily guessed $S^{\mu\nu\alpha\beta} \equiv \varepsilon^{\mu\nu\alpha\beta}$. $\varepsilon^{\mu\nu\alpha\beta}$ is the well-known totally antisymmetric $SO(4)$ invariant tensor. $\varepsilon_{\mu\nu\alpha\beta}$ is also called the Levi-Civita symbol. If $\lambda = \pm 1$ one recovers the well-known dual anti-self dual equations, for other values of λ the field strengths are trivial, i.e. $F_{\mu\nu} = 0$.

In dimensions greater than four $S^{\mu\nu\alpha\beta}$ is no longer under $SO(D)$. Given this the question arises if one could generalize in some manner the idea of self-duality in higher dimensions, i.e. $D > 4$. This issue was addressed by Corrigan et al. [6] who classified possible choices of $S^{\mu\nu\alpha\beta}$ upto eight dimensions subject to the condition that $T^{\mu\nu\alpha\beta}$ be invariant under maximal subgroup of $SO(D)$.

For example in eight-dimensions $D = 8$ the maximal subgroups of $SO(8)$ are $SU(3)/Z_3$, $SU(2) \otimes Sp(4)/Z_2$, $SO(5) \otimes SO(3)$, $SO(6) \otimes SO(2)$, $SO(7)$, $SO(4) \otimes SO(4)$, $SU(4) \times U(1)/Z_4$ and $\widehat{SO}(7)$ [6].

Recently the eight-dimensional case when the holonomy group in $SO(8)$ is either $SU(4)$ [the case of a Calabi-Yau four-fold] or $Spin(7)^{\dagger\dagger}$ [the case of a Joyce Manifold] has received attention [14,15] in the context of cohomological Yang-Mills theory in the said dimensions. In particular the case of Joyce Manifold seems to be special since it gives rise to octonionic structure. The eight dimensional tensors $S^{\mu\nu\alpha\beta}$ can be written in terms of the structure constants for octonions, i.e.,

$$\begin{aligned} S_{8ijk} &= c_{ijk}, \quad 1 \leq i, j, k \leq 7, \\ S_{lij k} &= \frac{1}{24} \varepsilon^{lijkabc} c_{abc}, \quad 1 \leq l, i, j, k \leq 7. \end{aligned} \quad (A2)$$

We note that octonions are natural generalization of the more familiar quaternions of four dimensions. Using Eq. A2 in the instanton equation Eq. A1 the field-strength in eight dimensions can be written as ^{‡‡}

$$\lambda F_{8i} = \frac{1}{2} c_{ijk} F_{jk}, \quad 1 \leq i, j, k \leq 7. \quad (A3)$$

If we explicitly write out Eq. A3 we obtain [6,14] a set of seven equations, for $\lambda = 1$ viz

^{**}We note that for the purposes of this appendix D refers to D -dimensional Euclidean space [6] and must not be confused with the D used in the main body of the paper.

^{††}We note that $Spin(7)$ is same as $\widehat{SO}(7)$

^{‡‡}Eq. A3 looks very similar to the four dimensional relation $F_{4i} = \frac{1}{2} \varepsilon_{ijk} F_{jk}$.

$$\begin{aligned}
F_{12} + F_{34} + F_{56} + F_{78} &= 0, \\
F_{13} + F_{42} + F_{57} + F_{86} &= 0, \\
F_{14} + F_{23} + F_{76} + F_{85} &= 0, \\
F_{15} + F_{62} + F_{73} + F_{48} &= 0, \\
F_{16} + F_{25} + F_{38} + F_{47} &= 0, \\
F_{17} + F_{82} + F_{35} + F_{64} &= 0, \\
F_{18} + F_{27} + F_{63} + F_{54} &= 0.
\end{aligned} \tag{A4}$$

A simple and instructive way of constructing $S^{\mu\nu\alpha\beta}$ and investigating its properties in eight dimensions in context of the maximal subgroup $\widehat{SO}(7)$ [i.e. $Spin(7)$ group] is to define it in terms of constant spinor η as

$$S^{\mu\nu\alpha\beta} = \eta^T \gamma^{\mu\nu\alpha\beta} \eta \tag{A5}$$

where $\gamma^{\mu\nu\alpha\beta}$ is defined to be totally antisymmetric product of γ matrices for $SO(8)$ spinor representations, viz

$$\gamma^{\mu\nu\alpha\beta} = \frac{1}{4!} \gamma^{[\mu} \gamma^\nu \gamma^\alpha \gamma^{\beta]}, \tag{A6}$$

and η is a constant unit spinor,

$$\eta^T \eta = 1. \tag{A7}$$

The γ matrices and spinors can be chosen to be real. Further one may choose a representation of the γ matrices which makes the decomposition of Eq. A5 into its two irreducible parts explicit [6]. To this end let

$$\gamma^8 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \lambda^i \\ -\lambda^i & 0 \end{pmatrix}, \quad i = 1, \dots, 7, \tag{A8}$$

where the 8×8 λ^i are antisymmetric and satisfy the usual relations

$$\{\lambda^i, \lambda^j\} = -2\delta^{ij}. \tag{A9}$$

We note that γ^9 is block-diagonal and so is $\gamma^{\mu\nu\alpha\beta}$

$$\begin{aligned}
\gamma^9 &= \prod_{i=1}^{i=8} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma^{8ijk} &= \begin{pmatrix} \lambda^i \lambda^j \lambda^k & 0 \\ 0 & -\lambda^i \lambda^j \lambda^k \end{pmatrix}, \quad i, j, k = 1, \dots, 7, \\
\gamma^{ijkl} &= \begin{pmatrix} \lambda^i \lambda^j \lambda^k \lambda^l & 0 \\ 0 & \lambda^i \lambda^j \lambda^k \lambda^l \end{pmatrix} \quad i, j, k, l = 1, \dots, 7.
\end{aligned} \tag{A10}$$

In Eq. A10 all the indices take on distinct values. The block-diagonal Eq. A10 form allows us to choose η to be left-handed or right-handed. The duality property of $S^{\mu\nu\alpha\beta}$ immediately follows since ${}^*\gamma^{\mu\nu\alpha\beta} = \gamma^9\gamma^{\mu\nu\alpha\beta}$.

The left-handed and right-handed parts of $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ are each complete set of 28 antisymmetric 8×8 matrices,

$$\begin{aligned}\gamma^{8i} &= \begin{pmatrix} -\lambda^i & 0 \\ 0 & \lambda^i \end{pmatrix}, \\ \gamma^{ij} &= \begin{pmatrix} -\frac{1}{2}\lambda^{[i}\lambda^{j]} & 0 \\ 0 & -\frac{1}{2}\lambda^{[i}\lambda^{j]} \end{pmatrix}, \\ \lambda_{IJ}^{ij}\lambda_{KL}^{ij} &= -8(\delta_{IJ}\delta_{KL} - \delta_{IK}\delta_{JL}), \quad \lambda^8 = 1, \quad i, j = 1, \dots, 8.\end{aligned}\tag{A11}$$

The eigenvalues λ of $S^{\mu\nu\alpha\beta}$ can be found from the completeness relation

$$\frac{1}{2}S_{\mu\nu\alpha\beta}S_{\alpha\beta\rho\sigma} + 2S_{\mu\nu\rho\sigma} - 3(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) = 0\tag{A12}$$

Contracting Eq. A12 with $F_{\alpha\beta}F_{\rho\sigma}$ and using Eq. A1 we get

$$\lambda^2 + 2\lambda - 3 = 0\tag{A13}$$

by demanding non-trivial relations among field strength components. Eq. A12 is easily solved and one obtains the solution $\lambda = 1, -3$.

We note that the adjoint representation of $SO(8)$ decomposes to $\underline{21} \oplus \underline{7}$ under any $SO(7)$ embeddings or alternatively the second rank tensor F_{ij} in eight dimensions belongs to $\underline{28}$ of $SO(8)$ which under $SO(7)$ decomposes as $\underline{28} = \underline{21} \oplus \underline{7}$. The eigenvalue $\lambda = 1$ corresponds to the $\underline{21}$ and yields the seven linear relations between the curvature or field-strength, viz Eq. A4. The other eigenvalue is $\lambda = -3$ and gives a set of 21 equations,

$$\begin{aligned}F_{12} &= F_{34} = F_{56} = F_{78}, \\ F_{13} &= F_{42} = F_{57} = F_{86}, \\ F_{14} &= F_{23} = F_{76} = F_{85}, \\ F_{15} &= F_{62} = F_{73} = F_{48}, \\ F_{16} &= F_{25} = F_{38} = F_{47}, \\ F_{17} &= F_{82} = F_{35} = F_{64}, \\ F_{18} &= F_{27} = F_{63} = F_{54}.\end{aligned}\tag{A14}$$

The octonion structure constants are invariant under G_2 and in this sense G_2 is the most interesting subgroup of $SO(7)$. As noted by Corrigan et al., [6] we can obtain the relevant $D = 7$ case simply by deleting terms or components with index 8 in A4 and A14, a set of 7 equations

$$\begin{aligned}
F_{12} + F_{34} + F_{56} &= 0, \\
F_{13} + F_{42} + F_{57} &= 0, \\
F_{14} + F_{23} + F_{76} &= 0, \\
F_{15} + F_{62} + F_{73} &= 0, \\
F_{16} + F_{25} + F_{47} &= 0, \\
F_{17} + F_{35} + F_{64} &= 0, \\
F_{27} + F_{63} + F_{54} &= 0,
\end{aligned} \tag{A15}$$

and a set of 14 equations,

$$\begin{aligned}
F_{12} &= F_{34} = F_{56}, \\
F_{13} &= F_{42} = F_{57}, \\
F_{14} &= F_{23} = F_{76}, \\
F_{15} &= F_{62} = F_{73}, \\
F_{16} &= F_{25} = F_{47}, \\
F_{17} &= F_{35} = F_{64}, \\
F_{27} &= F_{63} = F_{54}.
\end{aligned} \tag{A16}$$

The $D = 6$ case of interest can be obtained by deleting the quantities with 7 as a subscript in Eq. A15,

$$\begin{aligned}
F_{12} + F_{34} + F_{56} &= 0, \\
F_{13} + F_{42} &= 0, \\
F_{14} + F_{23} &= 0, \\
F_{15} + F_{62} &= 0, \\
F_{16} + F_{25} &= 0, \\
F_{35} + F_{64} &= 0, \\
F_{63} + F_{54} &= 0.
\end{aligned} \tag{A17}$$

Eq. A17 is a set of 7 equations, for the $\lambda = 1$ case appropriate for the $D = 6$ case. We recall that the relevant subgroup of $SO(6)$ is $SU(3) \otimes U(1)/Z_3$ [6]. Finally deleting terms with 5 and 6 in Eq. A17 takes us to the $\lambda = -1$ [anti-self dual case] of four-dimensional case.

REFERENCES

- [1] N. Ishibashi et al., Nucl. Phys. **B498** (1997) 467-491, hep-th/9612115.
- [2] A. Fayyazuddin et al., Nucl. Phys. **B499** (1997) 159-182, hep-th/9612115.
- [3] T. Banks et al., Phys. Rev. **D55** (1997) 5112, hep-th/9610043.
- [4] Y. Makeenko, hep-th/9704075, April 1997.
- [5] M. Claudson and M. B. Halpern, Nucl. Phys. **B250** (1996) 689-715.
- [6] E. Corrigan et al., Nucl. Phys. **B214** (1983) 465-480.
- [7] S. Hirano and M. Kato, hep-th/9708039, August 1997.
- [8] C. Hofman and J-S. Park, hep-th/9706130, June 1997.
- [9] E. Witten, Nucl. Phys. **B460** (1996) 335-350, hep-th/9510135.
- [10] J. Polchinski, hep-th/9510017, Oct. 1995.
- [11] M. Kaku, Strings, Conformal Fields and Topology, Springer-Verlag, 1991.
- [12] E. Witten, Commun. Math. Phys. **117** (1988) 353.
- [13] L. Baulieu I. M. Singer, Nucl. Phys. B (Proc. Suppl.) **6B** (1988) 12-19.
- [14] L. Baulieu et al., hep-th/9705127, May 1997; L. Baulieu et al., hep-th/9704167, April 1997.
- [15] B. S. Acharya et al., hep-th/9705138, June 1997.
- [16] C. Vafa, Nucl. Phys. **B469** (1996) 403, hep-th/9602122.
- [17] I. Ya. Are'eva, hep-th/9706146, June 1997.
- [18] T. Yoneya, Prog. Theor. Phys. **97**:949-962, 1997, hep-th/970378, March 1997.
- [19] M. Atiyah et al., Phys. Lett. **65A** (1978) 185.
- [20] S. K. Donaldson, Commun. Math. Phys. **93** (1984) 453.
- [21] S. Sethi and L. Susskind, hep-th/9702101, February 1997.
- [22] H. Sugawara, hep-th/9708029, August 1997.
- [23] S. Gukov, hep-th/9709138, September 1997.